J. DIFFERENTIAL GEOMETRY
37 (1993) 729-733

ON SMOOTH RANK-1 MAPPINGS OF BANACH SPACES ONTO THE PLANE

S. M. BATES

Abstract

For any separable infinite-dimensional Banach space E we construct a surjective C^∞ mapping $f: E \to \mathbb{R}^2$ satisfying $\operatorname{rank} Df(v) \le 1$ for all $v \in E$.

A Fréchet differentiable map $f: E \to F$ is called *rank-r* provided $\operatorname{rank} Df(v) \le r$ for al $v \in E$. Surjective rank-1 mappings $f: \mathbb{R}^n \to \mathbb{R}^m$ are known to exist whenever $n > m > 1$ (see [1], [2], [6], [15]); by the classical Morse-Sard theorem, however, such mappings[1] cannot belong to the smoothness class C^{n-m+1}.

Let E denote a separable infinite-dimensional Banach space. The aim of this note is to construct a C^∞ rank-1 mapping of E onto $\mathbb{R}^2$. Because our technique generalizes easily to produce smooth rank-1 mappings of E onto any higher-dimensional Euclidean space, this settles a recent question of H. Sussmann [14] and Y. Yomdin [15] (see also [4, p. 59]).

To begin our construction, we recall that by a result of Johnson and Rosenthal [5] every separable infinite-dimensional Banach space has a quotient with a Schauder basis.[2] For our purposes, we may therefore assume that E has a bounded basis with corresponding unit coordinate functions $\{\lambda_j\}$ (cf. [11, p. 20f]). The symbol m_k denotes a $k \times k$ matrix with ij-entry $m_k(i, j) \in \{1, 3, 5, 7\}$, and the notation $m_k \prec m_l$ implies $m_k(i, j) = m_l(i, j)$ for $i, j = 1, \cdots, k$.

Cylinder Sets in E

Let $I(a_1, \cdots, a_k)$ denote the set of those $x \in [0, 1]$ such that a_i is the ith digit in the base-9 expansion of x. We define the family $\mathscr{B}$ of

Received June 17, 1991 and, in revised form, April 27, 1992. The author was supported by an NSF graduate fellowship in Mathematics.

[1]For a sharper smoothness bound in the context of singular mappings, see [1], [2].

[2]I am indebted to Y. Benyamini for calling the article [5] to my attention. An analogous construction can be carried out using the biorthogonal sequences constructed in [8], [9].

cylinder sets in E as the collection of all sets of the form

$$B(m_k) = \{v \in E: 9^i\lambda_i(v) \in I(m_k(1, i), \cdots, m_k(k, i)),\ i = 1, \cdots, k\}$$

for some m_k. The cylinder set $B(m_k)$ consists of those $v \in E$ whose first k coordinates lie in certain subintervals of $[0, 1]$ determined by the matrix m_k: For each $i = 1, \cdots, k$, the ith column of m_k comprises the first k digits in the base-9 expansion of $9^i\lambda_i(v)$. For fixed k, there are thus 4^{k^2} distinct $B(m_k)$, and by construction each $B(m_k)$ contains the 4^{2k+1} cylinder subsets $B(m_{k+1})$ for which $m_k \prec m_{k+1}$. If $l \geq k$ and m_k, m'_l are distinct, then for any $v \in \partial B(m_k)$ there exists $j \leq k$ such that $|\lambda_j(v - v')| \geq 9^{-(k+j+1)}$ for all $v' \in B(m'_l)$.

Since the chosen basis of E is bounded, the preceding definition implies that the set $\bigcap_k B(m_k)$ consists of a unique vector for any chain of matrices $\{m_k\}$. We define Λ to be the Cantor set defined by $\mathscr{B}$, i.e., the set of those $v \in E$ contained in infinitely many members of $\mathscr{B}$.

Mapping of Λ

Let R_0 be any closed square in $\mathbb{R}^2$. For each $k \in \mathbb{Z}^+$, we divide R_0 with lines parallel to its edges into 4^{k^2} congruent, closed subsquares $R(m_k) \subset R_0$ of diameter $M \cdot 2^{-k^2}$, and we require that our labelling is such that each $R(m_k)$ contains the 4^{2k+1} squares $R(m_{k+1})$ for which $m_k \prec m_{k+1}$. For each m_k, choose a point $p(m_k) \in R(m_k)$.

We define the map f on Λ by requiring that $f(\Lambda \cap B(m_k)) \subset R(m_k)$. Since for any $x \in R_0$ there exists a (possibly nonunique) chain of matrices m_k satisfying $\bigcap_k R(m_k) = \{x\}$, it follows that $R_0 \subset f(\Lambda)$. Moreover, if $v, v' \in \Lambda$ and $k \geq 2$ is the largest integer such that $v, v' \in B(m_{k-1})$, then $|v - v'| \geq 9^{-3k}$, and

$$|f(v) - f(v')| \leq M \cdot 2^{-k^2} \leq M \cdot |v - v'|^{k/12}.$$

Extension of f

Choose a smooth function $\varphi: \mathbb{R} \to \mathbb{R}$ such that $\varphi = 1$ on a neighborhood of $[-\frac{1}{2}, \frac{1}{2}]$ and $\varphi(x) = 0$ when $|x| \geq \frac{1}{2} + 9^{-2}$. Define $h_k: E \to \mathbb{R}$ by

$$h_k(v) = \prod_{j=1}^{k} \varphi(9^{k+j}\lambda_j(v)).$$

Clearly the function h_k is smooth; since the linear map $E \to \mathbb{R}^k$ given by $v \mapsto (\lambda_1(v), \cdots, \lambda_k(v))$ has norm $\leq \sqrt{k}$, a simple computation shows furthermore that $\|D^n h_k\| \leq M_n^k$ for each $n, k \in \mathbb{Z}^+$ and constants M_n independent of k.

We now fix m_k and let $m_{k+1,i}$ denote the 4^{2k+1} immediate successors of m_k. Consider $B = B(m_k)$ and its cylinder subsets $B_i = B(m_{k+1,i})$. For each i, there exists a translation $T_i: E \to E$ which maps B_i onto $\{v \in E: 2|\lambda_j(v)| \leq 9^{-(k+j+1)},\ j = 1, \cdots, k+1\}$. Defining $g_i: E \to \mathbb{R}$ as the composition $h_{k+1} \circ T_i$, we observe the following:

(1) $g_i = 1$ on a neighborhood of B_i.

(2) $\operatorname{Supp}(g_i) \subset \operatorname{Int} B$, and $\operatorname{Supp}(g_i) \cap \operatorname{Supp}(g_j) = \varnothing$ whenever $i \neq j$.

(3) $\|D^n g_i\| \leq M_n^{k+1}$ for all $n \in \mathbb{Z}^+$, $i = 1, \cdots, 4^{2k+1}$.

We now define the partial extension of f to the region $B \setminus \bigcup B_i$ by

$$f = p + \sum_{i=1}^{4^{2k+1}} (p_i - p) g_i,$$

where $p = p(m_k)$, $p_i = p(m_{k+1,i})$. Analogously f is extended to $E \setminus \bigcup B(m_1)$.

Smoothness of f

By condition (1) and the preceding definition, it follows that f is a continuous extension of our mapping of Λ. Since $D^n f = 0$ on the boundary of each cylinder set, the map f is C^∞ on $E \setminus \Lambda$.

To determine the smoothness of f at points of Λ, we first note that by conditions (2) and (3) above,

$$\|D^n f\| \leq M_n^{k+1} \cdot \operatorname{diam}(R(m_k)) = M \cdot M_n^{k+1} \cdot 2^{-k^2}$$

on $B(m_k) \setminus \bigcup_i B(m_{k+1,i})$; thus $\|D^n f\|$ tends to zero on approach to Λ for all $n \in \mathbb{Z}^+$. Recalling our previous estimate for the modulus of continuity of $f|_\Lambda$, we conclude that f is C^∞ on E by inductively applying the following fact whose proof is left to the interested reader.

Lemma. *Let X, Y be Banach spaces, $A \subset X$ a closed subset, and $g: X \to Y$ a continuous map, differentiable on $X \setminus A$. If $x \in A$ and*

(a) $|g(x) - g(z)| = o(|x - z|)$ *as* $z \to x$, $z \in A$,
(b) $\|Dg(z')\| = o(1)$ *as* $z' \to x$, $z' \in E \backslash A$,
then g *is differentiable at* x *and* $Dg(x) = 0$.

From the above remarks it follows in particular that our mapping $f: E \to \mathbb{R}^2$ satisfies $Df = 0$ on the Cantor set Λ, and thus $\operatorname{rank} Df(v) = 0$ for all $v \in \Lambda$. On the complement of Λ, condition (2) implies that f is locally of the form $f = wg + w'$ for some smooth function g and vectors $w, w' \in \mathbb{R}^2$; consequently, $\operatorname{rank} Df(v) \leq 1$ for all $v \in E \backslash \Lambda$, and so f is rank-1.

In order to map E onto $\mathbb{R}^2$, we choose a sequence $\{\mathscr{B}_i\}$ of distinct cylinder set families in E, requiring that any two members of different families be separated by a distance ≥ 1. By the above construction, there exists for each $i \in \mathbb{Z}^+$ a smooth rank-1 mapping of E onto the square $[-i, i]^2$ which equals $(0, 0)$ outside $\bigcup_{\mathscr{B}_i} B$. Piecing these mappings together then produces the desired smooth rank-1 surjection $E \to \mathbb{R}^2$.

Remarks

An important observation regarding the Cantor set Λ is that it cannot be the countable union of sets having finite Hausdorff dimension. To prove this statement, we recall the following weak infinite-dimensional version of the Morse-Sard theorem from [2] (compare [3, Theorem 3.4.3], [10], [12]):

Theorem. *Let* X, Y *be separable Banach spaces,* $A \subset X$ *a set of Hausdorff dimension* $s_0 < \infty$, *and* $f: X \to Y$ *a* C^p *map satisfying* $D^k f(x) = 0$ *for each* $x \in A$, $k = 1, 2, \cdots, p$. *Then the Hausdorff dimension of* $f(A)$ *is at most* s_0/p.

As noted previously, the map $f: E \to \mathbb{R}^2$ constructed above satisfies $D^n f(x) = 0$ for all $x \in \Lambda$, $n \in \mathbb{Z}^+$. Thus, if $A \subset \Lambda$ has finite Hausdorff dimension, its image $f(A)$ has Hausdorff dimension zero. Since $f(\Lambda)$ has nonempty interior, our assertion follows.

Some questions

In view of the preceding remarks, it would be interesting to determine precisely how large a set $A \subset E$ must be in order that its image under *some* smooth rank-1 mapping into the plane has nonempty interior. We conclude our discussion with two specific questions illustrating this point:

1. Does there exist a C^∞ rank-1 map $f: E \to \mathbb{R}^2$ such that $f(A)$ has nonempty interior for some subset $A \subset E$ of *finite* Hausdorff dimension?

Note that by the preceding theorem, any such set on which $Df = 0$ must have dimension ≥ 2. A dual question suggested by our construction concerns necessary restrictions on the size and geometry of the target space.

2. Does every separable, infinite-dimensional Banach space E admit a C^{∞} rank-1 mapping onto every separable Banach space F?

We hope to return to these points in a sequel to this paper.

References

[1] S. M. Bates, *On the image size of singular maps*. I, Proc. Amer. Math. Soc. **114** (1991) 699–705.

[2] ——, *On the image of size of singular maps*. II, Duke Math. J. **68** (1992), 463–476.

[3] H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., Vol. 153, Springer, New York, 1969.

[4] W. H. Fleming, *Future directions in control theory: a mathematical perspective*, Report of the Panel on Future Directions in Control Theory, SIAM Reports, Philadelphia, 1988.

[5] W. B. Johnson & H. P. Rosenthal, *On w^*-basic sequences and their applications to the study of Banach spaces*, Studia Math. **43** (1972) 77–92.

[6] R. Kaufman, *A singular map of a cube onto a square*, J. Differential Geometry **14** (1979) 593–594.

[7] A. P. Morse, *The behavior of a function on its critical set*, Ann. of Math. (2) **40** (1939) 62–70.

[8] R. I. Ovsepian & A. Pełczynski, *The existence in every separable Banach space of a fundamental total and bounded biorthogonal system*, Studia Math. **54** (1975) 149–159.

[9] A. Pełczynski, *any separable Banach space admits for every $\varepsilon > 0$ fundamental and total biorthogonal sequences bounded by $1 + \varepsilon$*, Studia Math. **55** (1976) 295–304.

[10] A. Sard, *Images of critical sets*, Ann. of Math (2) **68** (1958) 247–259.

[11] I. Singer, *Bases in Banach spaces*. I, Grundlehren Math. Wiss., Vol. 154, Springer, New York, 1970.

[12] S. Smale, *An infinite dimensional version of Sard's Theorem*, Amer. J. Math. **87** (1965) 861–866.

[13] S. Sternberg, *Lectures on differential geometry*, Prentice -Hall, Englewoods Cliffs, NJ, 1964.

[14] H. Sussmann, private communication, May 1991.

[15] Y. Yomdin, *Surjective mappings whose differential is nowhere surjective*, Proc. Amer. Math. Soc. **111** (1991) 267–270.

UNIVERSITY OF CALIFORNIA, BERKELEY